Relaxation of the distribution function tails for systems described by Fokker-Planck equations

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We study the formation and the evolution of velocity distribution tails for systems with weak long-range interactions. In the thermal bath approximation, the evolution of the distribution function of a test particle is governed by a Fokker-Planck equation where the diffusion coefficient depends on the velocity. We extend the theory of Potapenko *et al.* [Phys. Rev. E **56**, 7159 (1997)] developed for power-law diffusion coefficients to the case of an arbitrary form of diffusion coefficient and friction force. We study how the structure and the progression of the front depend on the behavior of the diffusion coefficient and friction force for large velocities. Particular emphasis is given to the case where the velocity dependence of the diffusion coefficient is Gaussian. This situation arises in Fokker-Planck equations associated with one dimensional systems with long-range interactions such as the Hamiltonian mean field (HMF) model and in the kinetic theory of two-dimensional point vortices in hydrodynamics. We show that the progression of the front is extremely slow (logarithmic) in that case so that the convergence towards the equilibrium state is peculiar. Our general formalism can have applications for other physical systems such as optical lattices.

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I. INTRODUCTION

The study of Fokker-Planck equations is an important problem in statistical mechanics and kinetic theory [1]. In the simplest models, the diffusion coefficient is constant. However, Fokker-Planck equations with a diffusion coefficient depending on the velocity of the particles have also been introduced in physics. These equations usually describe the relaxation of a "test particle" evolving in a bath of "field particles" at statistical equilibrium when the particles interact via weak long-range forces. In that case, the diffusion coefficient is a function of the velocity of the test particle. For example, in his Brownian theory of stellar dynamics, Chandrasekhar [2] describes the evolution of the velocity distribution of a star in a cluster by a Fokker-Planck equation involving a diffusion and a friction. The coefficients of diffusion and friction are related to each other by an Einstein relation and the diffusion coefficient decreases as v^{-3} for large velocities. These results are similar to those obtained in plasma physics for the Coulombian interaction [3]. By using an analogy with stellar dynamics, Chavanis [4–6] describes the relaxation of a test vortex in a thermal bath of field vortices by a Fokker-Planck equation (in position space) involving a diffusion and a drift along the vorticity gradient. The coefficients of drift and diffusion are related to each other by a form of Einstein relation involving a negative temperature and the diffusion coefficient is inversely proportional to the local shear created by the vortex cloud. For a Gaussian distribution of field vortices, the diffusion coefficient of the test vortex decreases with the distance as $r^2e^{-\lambda r^2}$ [7,8]. Similarly, for the Hamiltonian mean field (HMF) model, Bouchet and Dauxois [9,10], and Chavanis *et al.* [11] find that the velocity distribution of a test particle satisfies a Fokker-Planck equation with a diffusion coefficient decreasing as $e^{-\beta v^2/2}$ for large velocities. More generally, using the theory developed by Landau, Lenard, and Balescu in plasma physics [12] and implementing a thermal bath approximation, one can obtain a general Fokker-Planck equation involving an anisotropic diffusion coefficient depending on the velocity of the test particle. This Fokker-Planck equation is valid for systems with weak long-range potentials of interaction. The preceding kinetic equations can be recovered as particular cases of this general Fokker-Planck equation [7].

For Fokker-Planck equations with a variable diffusion coefficient, the relaxation towards the Boltzmann distribution is slowed down, especially if the diffusion coefficient decreases rapidly with the velocity. One consequence is that velocity correlation functions can decrease *algebraically* rapidly with time (instead of exponentially) as investigated by Bouchet and Dauxois [10] in relation with the HMF model.[1] They therefore explain the observed algebraic tails of the velocity correlations functions in terms of classical kinetic theory without invocating a notion of "generalized thermodynamics." Here, we consider the relaxation of the system towards equilibrium from another point of view. We focus on the distribution function $f(v,t)$ and study the structure and the evolution of the front formed in the high velocity tail. Our study is based on the approach of Potapenko *et al.* [15] who

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[1]In that case, the Fokker-Planck equation with Gaussian diffusion coefficient and linear friction can be transformed into a Fokker-Planck equation with constant diffusion coefficient and logarithmic potential which is known to exhibit power-law correlations (see, in particular, Appendix B of Refs. [13] and [14]).

studied this problem in the case where the diffusion coefficient decreases algebraically with the velocity. In the present paper, we generalize this approach to an arbitrary form of diffusion coefficient and study how the front position $v_f(t)$ and the front structure evolve with time. In the case of a Gaussian or exponential decay of the diffusion coefficient with the velocity, we find that the progression of the front is extremely slow (logarithmic). This can lead to a sort of "kinetic blocking" or, at least, a "slowing down" of the relaxation. A similar confining effect was noted in the case where the diffusion coefficient depends on the density [16,17], but this situation (related to the theory of violent relaxation) is more difficult to analyze since the kinetic equation is then nonlinear.

The paper is organized as follows. In Sec. II, we discuss different systems with long-range interactions (stellar systems, Coulombian plasmas, two-dimensional vortices, HMF model, etc.) which are described in the thermal bath approximation by Fokker-Planck equations with a variable diffusion coefficient. We also consider systems like optical lattices described by Fokker-Planck equations with constant diffusion coefficient but variable friction coefficient. In Sec. III, we generalize the theory of Potapenko *et al.* [15] for an arbitrary form of diffusion coefficient and friction force. We provide general equations characterizing the structure and the evolution of the front for large times. In Sec. IV, we address the validity of our approach. In Sec. V, we consider particular applications of our general formalism for physically motivated Fokker-Planck equations. Analytical results are compared with direct numerical simulations of the Fokker-Planck equation. Particular emphasis is given to the case where the diffusion coefficient decreases with the velocity like an exponential or a Gaussian distribution. This is the situation relevant for one-dimensional systems like the HMF model and for two-dimensional point vortices. Finally, in Sec. VI we investigate a class of Fokker-Planck equations for which our approach is exact for all times. The Appendixes provide technical details and extensions of our main results.

II. EXAMPLES OF FOKKER-PLANCK EQUATIONS WITH A VARIABLE DIFFUSION COEFFICIENT AND FRICTION FORCE

We consider a Hamiltonian system of N particles interacting via a weak long-range binary potential $u(|\mathbf{r}-\mathbf{r}'|)$. These particles can be stars in stellar clusters, electrons or ions in a plasma, point vortices in two-dimensional hydrodynamics, particles located on a ring in the HMF model, etc. We assume that the cluster is homogeneous and in a steady state characterized by a distribution function $f_0(\mathbf{v})$. In general, f_0 will be the statistical equilibrium state (thermal bath) but in certain cases it can be a slowly evolving distribution function. We introduce a "test particle" and denote by $P(\mathbf{v},t)$ its velocity distribution function. Due to the interaction with the "field particles," the velocity distribution will change and the test particle will acquire the distribution of the bath $f_0(\mathbf{v})$ for $t\to+\infty$ (see [7] for more details). The general Fokker-Planck equation describing the relaxation ("thermalization") of the test particle in the bath of field particles can be written as [7]

$$\frac{\partial P}{\partial t}=\pi(2\pi)^d m\frac{\partial}{\partial v^\mu}\int d\mathbf{v}_1 d\mathbf{k}k^\mu k^\nu\frac{\hat{u}(\mathbf{k})^2}{|\epsilon(\mathbf{k},\mathbf{k}\cdot\mathbf{v})|^2}\delta(\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}_1))\times\left(\frac{\partial}{\partial v^\nu}-\frac{\partial}{\partial v_1^\nu}\right)f_0(\mathbf{v}_1)P(\mathbf{v},t), \quad (1)$$

where $\hat{u}(\mathbf{k})$ is the Fourier transform of $u(\mathbf{r})$ and

$$\epsilon(\mathbf{k},\omega)=1+(2\pi)^d\hat{u}(\mathbf{k})\int\frac{\mathbf{k}\cdot\frac{\partial f_0}{\partial\mathbf{v}}}{\omega-\mathbf{k}\cdot\mathbf{v}}d\mathbf{v}, \quad (2)$$

is the dielectric function. This equation can be obtained from the Lenard-Balescu equation by *fixing* the distribution function of the field particles $f(\mathbf{v}_1,t)$ to its static value $f_0(\mathbf{v}_1)$ [3,7]. This (thermal) bath approximation transforms an integro-differential equation (Lenard-Balescu) into a differential equation (Fokker-Planck). The Lenard-Balescu equation was introduced in plasma physics for the Coulombian potential but it can apply to other systems with weak long-range interactions. Equations (1) and (2) can also be obtained from the general expression of the Fokker-Planck equation by explicitly calculating the coefficients of friction and diffusion (first and second moments of the velocity increments) using the Klimontovich approach [12]. The fact that Eq. (1) is linear does not imply that the distribution $P(\mathbf{v},t)$ is close to equilibrium. The test particle approach is different from considering a small perturbation of the Lenard-Balescu equation around equilibrium. In the first case, we describe the evolution of a single test particle (or an ensemble of test particles that do not interact among themselves) in a thermal bath while in the second case one would describe the evolution of all the particles (the system "as a whole") close to equilibrium.

If we consider that the field particles are at statistical equilibrium with the Boltzmann distribution $f_0(\mathbf{v})\sim e^{-\beta m v^2/2}$, and if we neglect collective effects taking $|\epsilon(\mathbf{k},\mathbf{k}\cdot\mathbf{v})|=1$ (Landau approximation) we can rewrite the general Fokker-Planck equation (1) in the form [7]

$$\frac{\partial P}{\partial t}=\frac{1}{t_R}\frac{\partial}{\partial x^\mu}\left[G^{\mu\nu}(\mathbf{x})\left(\frac{\partial P}{\partial x^\nu}+2Px^\nu\right)\right], \quad (3)$$

where we have set $\mathbf{x}=(\beta m/2)^{1/2}\mathbf{v}$. The diffusion coefficient is proportional to the tensor

$$G^{\mu\nu}(\mathbf{x})=\int d\hat{\mathbf{k}}\hat{k}^\mu\hat{k}^\nu e^{-(\hat{\mathbf{k}}\cdot\mathbf{x})^2} \quad (4)$$

with $\hat{\mathbf{k}}=\mathbf{k}/k$, and the quantity

$$t_R^{-1}=\left(\frac{\pi}{8}\right)^{1/2}d^{3/2}(2\pi)^d\frac{\rho m}{v_m^3}\int_0^{+\infty}k^d\hat{u}(k)^2dk \quad (5)$$

provides an estimate of the inverse relaxation time of the test particle toward the distribution of the bath. Here $v_m=(d/\beta m)^{1/2}$ denotes the rms velocity and ρ the spatial density. A more general expression of the diffusion coefficient can be obtained by taking into account collective effects [7]. Note, however, that for $v\to\infty$ (which is a limit that we shall be particularly interested with in the sequel), the expression

(4) is asymptotically exact since $|\epsilon(\mathbf{k},\mathbf{k}\cdot\mathbf{v})|\to 1$ for $|\mathbf{v}|\to+\infty$. Note also that for weak long-range potentials of interaction for which our approach is valid, and in the Landau approximation, the precise form of the potential $\hat{u}(k)$ only determines the time scale of the relaxation, through Eq. (5), not the form of the kinetic operator. Therefore, in the Landau approximation, the expression of the diffusion tensor as a function of the velocity only depends on the dimension of space d.

In dimension $d=3$, the diffusion tensor can be written

$$G^{\mu\nu}=\left(G_{\parallel}-\frac{1}{2}G_{\perp}\right)\frac{x^{\mu}x^{\nu}}{x^2}+\frac{1}{2}G_{\perp}\delta^{\mu\nu}, \tag{6}$$

where $G_{\parallel}$ and $G_{\perp}$ are the diffusion coefficients in the directions parallel and perpendicular to the velocity of the test particle. They are explicitly given by

$$G_{\parallel}=\frac{2\pi^{3/2}}{x}G(x), \tag{7a}$$

$$G_{\perp}=\frac{2\pi^{3/2}}{x}[\mathrm{erf}(x)-G(x)], \tag{7b}$$

with

$$G(x)=\frac{2}{\sqrt{\pi}}\frac{1}{x^2}\int_0^x t^2e^{-t^2}dt=\frac{1}{2x^2}\left[\mathrm{erf}(x)-\frac{2x}{\sqrt{\pi}}e^{-x^2}\right], \tag{8}$$

where

$$\mathrm{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt, \tag{9}$$

is the error function. If we consider spherically symmetric distributions, noting that $\partial P/\partial x^{\mu}=(1/x)(\partial P/\partial x)x^{\mu}$ and $G^{\mu\nu}x^{\nu}=G_{\parallel}x^{\mu}$ we obtain

$$\frac{\partial P}{\partial t}=\frac{1}{t_R}\frac{1}{x^2}\frac{\partial}{\partial x}\left[x^2G_{\parallel}(x)\left(\frac{\partial P}{\partial x}+2Px\right)\right]. \tag{10}$$

For the gravitational potential, this Fokker-Planck equation has been studied by Chandrasekhar in his Brownian theory of stellar dynamics [2]. It has also been considered in plasma physics as an approximation of the Landau equation valid for sufficiently large times [3,15]. We note in particular that the diffusion coefficient $G_{\parallel}(x)$ decreases algebraically like x^{-3} for $x\to+\infty$.

Alternatively, if we consider one dimensional systems ($d=1$), the general Fokker-Planck equation (1) simplifies into [7]

$$\frac{\partial P}{\partial t}=\frac{\partial}{\partial v}\left[D(v)\left(\frac{\partial P}{\partial v}-P\frac{d}{dv}\ln f_0\right)\right], \tag{11}$$

where $D(v)$ is given by

$$D(v)=4\pi^2mf_0(v)\int_0^{+\infty}dk\frac{k\hat{u}(k)^2}{|\epsilon(k,kv)|^2}. \tag{12}$$

We note that the distribution function $P(v,t)$ of the test particle relaxes towards the distribution of the bath $f_0(v)$ on a time scale of the order Nt_D, where t_D is the dynamical time [7]. If we neglect collective effects, or consider the limit of large velocities, we find that the diffusion coefficient is given by

$$D(v)=4\pi^2mf_0(v)\int_0^{+\infty}dkk\hat{u}(k)^2. \tag{13}$$

It is proportional to the distribution function of the bath $f_0(v)$. In particular, if the field particles are at statistical equilibrium with a Gaussian distribution, the diffusion coefficient decreases like $e^{-\beta mv^2/2}$. This type of Fokker-Planck equations apply for example to the HMF model [9–11] which can be viewed as the one Fourier component of a one-dimensional plasma (or self-gravitating system) [11]. More generally, these Fokker-Planck equations (11) are valid for a wide class of one-dimensional systems with weak long-range interactions [7]. We note that for one-dimensional systems the Lenard-Balescu collision term cancels out so that the distribution function of the field particles $f(v,t)$ does not evolve, i.e., $\partial f/\partial t=0$, on a time scale of order Nt_D. Since, on the other hand, the relaxation time of a test particle toward the distribution of the bath is of order Nt_D, this implies that we can assume that the distribution of the field particles is stationary $f(v,t)=f_0(v)$ when we study the relaxation of a test particle. This is true for *any* distribution function $f_0(v)$ that is a stable stationary solution of the Vlasov equation [7,10,11]. This is not true in higher dimensions $d=2$ and $d=3$, except for the Maxwellian distribution $f_e(\mathbf{v})$, since the distribution of the field particles $f(\mathbf{v},t)$ changes on a time Nt_D as it relaxes towards the statistical equilibrium state $f_e(\mathbf{v})$. Furthermore, even if we assume that the distribution of the bath $f_0(\mathbf{v})$ is approximately stationary, the distribution of the test particle $P(\mathbf{v},t)$ will not relax towards $f_0(\mathbf{v})$ for large times except if $f_0(\mathbf{v})$ is Maxwellian [7].

A Fokker-Planck equation with a space dependent diffusion coefficient has been introduced by Chavanis in [4–6] to describe the relaxation of a test vortex in a "sea" of field vortices with vorticity profile $\omega_0(r)$. For an axisymmetric distribution, the Fokker-Planck equation for $P(r,t)$ can be written [5,7]

$$\frac{\partial P}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}\left[rD(r)\left(\frac{\partial P}{\partial r}-P\frac{d}{dr}\ln\omega_0\right)\right], \tag{14}$$

with a diffusion coefficient

$$D(r)=\frac{\gamma}{8}\frac{1}{|\Sigma(r)|}\ln N\omega_0(r), \tag{15}$$

where $\Sigma(r)=r\Omega_0'(r)$ is the local shear created by the field vortices [$\Omega_0(r)$ represents the angular velocity related to the vorticity by $\omega_0(r)=(1/r)(r^2\Omega_0)'$]. For a vorticity profile $\omega_0(r)=Ae^{-\lambda r^2}$ of the field vortices, it is easy to see that the diffusion coefficient of the test vortex decreases like $D(r)\sim r^2e^{-\lambda r^2}$ for $r\to+\infty$ [7,8].

Finally, Fokker-Planck equations with diffusion and friction coefficients depending on the velocity can occur in many areas of physics. For example, the motion of atoms in

a one-dimensional optical lattice formed by two counter-propagating laser beams with linear perpendicular polarization can be described, after spatial averaging, by a Fokker Planck equation of the form [13,14]

$$\frac{\partial W}{\partial t} = \frac{\partial}{\partial p}\left[D(p)\frac{\partial W}{\partial p} - WK(p) \right], \tag{16}$$

with

$$K(p) = -\frac{\alpha p}{1+(p/p_c)^2}, \quad D(p) = D_0 + \frac{D_1}{1+(p/p_c)^2}. \tag{17}$$

For $p \to +\infty$, $D(p) \to D_0$ and $K(p) \sim -\alpha p_c^2/p$. This corresponds to a logarithmic potential $U(p) \sim (\alpha p_c^2/D_0)\ln p$ defined by $K(p)/D(p) = -U'(p)$. Equation (16) belongs to the general class of Fokker-Planck equations that we shall study in the sequel. The case of a logarithmic potential is treated specifically in Sec. V C.

III. GENERAL SOLUTION OF THE PROBLEM

The various examples discussed previously prompt us to study Fokker-Planck equations with a diffusion coefficient and friction force depending on the velocity. In particular, we can wonder how the distribution function $f(v,t)$ approaches the equilibrium distribution. This problem has been investigated by Potapenko *et al.* [15] in the case of 3D plasmas where the diffusion coefficient is a power law. These authors found that the asymptotic behavior of the velocity distribution tail has a propagating wave appearance. The high velocity tail develops a front at which the distribution function drops to zero. This front $v_f(t)$ progresses with time and goes to $v_f(t) \to +\infty$ for $t \to +\infty$. The profile of the front also deforms itself as time goes on. However, if we use an appropriate system of coordinates, it can be expressed in terms of the error function. We shall here formulate the problem in a general setting, for an arbitrary form of diffusion coefficient and friction force, and we shall investigate how the evolution of the front and the profile of the high velocity tail distribution depend on the form of the diffusion coefficient.

Let us consider the Fokker-Planck equation

$$\frac{\partial f}{\partial t} = \frac{1}{v^{d-1}}\frac{\partial}{\partial v}\left[v^{d-1}D(v)\left(\frac{\partial f}{\partial v} + fU'(v) \right) \right], \tag{18}$$

where $D(v) \geqslant 0$ and $U(v)$ are arbitrary functions. If the following zero flux condition

$$v^{d-1}D(v)\left(\frac{\partial f}{\partial v} + fU'(v) \right) \to 0 \tag{19}$$

when $v \to \infty$ is fulfilled, then the stationary solutions of this Fokker-Planck equation take the form

$$f_e(v) = Ae^{-U(v)}, \tag{20}$$

where A is a constant (normalization). The Fokker-Planck equation (18) decreases the free energy $F = E - S$ where $E = \int fU(v)d\mathbf{v}$ is the energy and $S = -\int f \ln f d\mathbf{v}$ is the Boltzmann entropy (the temperature has been included in the potential U). Indeed, one has

$$\dot{F} = -\int \frac{D}{f}\left(\frac{\partial f}{\partial \mathbf{v}} + f\frac{\partial U}{\partial \mathbf{v}} \right)^2 d\mathbf{v} \leqslant 0. \tag{21}$$

Therefore, if F is bounded from below, the distribution will converge towards the equilibrium state (20) for $t \to +\infty$. We want to analyze the propagation of the front in the high velocity tail of the distribution function. Thus, we set

$$f(v,t) = f_e(v)u(v,t). \tag{22}$$

For sufficiently large times, the core of the distribution function will have reached its asymptotic value (20) so that $u \simeq 1$ in that region. On the other hand, for sufficiently large velocities, the distribution has not relaxed yet and $u = 0$. Therefore, we expect the formation of a front at a typical velocity value $\sim v_f(t)$ where the function $u(v,t)$ passes from $u = 1$ to $u = 0$. On this phenomenological basis, $u(v,t)$ is the relevant function to consider in our "traveling front" analysis. Its exact evolution is governed by

$$\frac{\partial u}{\partial t} = \frac{1}{v^{d-1}}\frac{\partial}{\partial v}\left(v^{d-1}D(v)\frac{\partial u}{\partial v} \right) - D(v)U'(v)\frac{\partial u}{\partial v}, \tag{23}$$

which is obtained from Eqs. (18)–(22). If we perform the change of variables $dx/dv = 1/\sqrt{D(v)}$, we obtain the equivalent equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \left[\frac{1}{2}\frac{D'(v)}{\sqrt{D(v)}} + \frac{d-1}{v}\sqrt{D(v)} - U'(v)\sqrt{D(v)} \right]\frac{\partial u}{\partial x}, \tag{24}$$

where $v = v(x)$ must be viewed as an implicit function of x. If we introduce the velocity field

$$V(v) = \sqrt{D(v)}\left[U'(v) - \frac{d-1}{v} - \frac{1}{2}(\ln D)'(v) \right], \tag{25}$$

or, equivalently,

$$V(v) = -\sqrt{D(v)}\frac{d}{dv}\{\ln[v^{d-1}e^{-U(v)}D^{1/2}(v)]\}, \tag{26}$$

Eq. (24) can be rewritten as

$$\frac{\partial u}{\partial t} + V(v)\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}. \tag{27}$$

The structure of this equation is clear. The right-hand side corresponds to a diffusion and the left-hand side corresponds to an advection by a velocity field $V(v)$. This velocity field (in velocity space) governs the evolution of the front. To get a physical insight into the problem, let us first neglect the diffusion term. The resulting equation

$$\frac{\partial u}{\partial t} + V(v)\frac{\partial u}{\partial x} = 0 \tag{28}$$

can be solved with the method of characteristics. Writing

$$\frac{dx}{dt} = V(v) = \frac{dx}{dv}\frac{dv}{dt} = \frac{1}{\sqrt{D(v)}}\frac{dv}{dt}, \tag{29}$$

we get

$$\frac{dv}{dt} = \sqrt{D(v)}V(v) = D(v)\left[U'(v) - \frac{d-1}{v} - \frac{1}{2}(\ln D)'(v)\right]. \tag{30}$$

This equation determines the evolution of the front $v_f(t)$. In the extreme approximation where the diffusion is neglected, the profile of the front is given by a step function $u(x,t) = \eta(x - x_f(t))$ where η is the Heaviside function [$\eta(x)=1$ for $x<0$ and $\eta(x)=0$ for $x>0$]. As we shall see, the diffusion term will smooth out this profile.

To take into account the effect of diffusion, we return to Eq. (27) and perform the change of variables

$$z = x - x_f(t), \quad u(x,t) = \phi(z,t), \tag{31}$$

where the function $x_f(t)$ is *defined* by Eq. (29). Substituting this in Eq. (27), we obtain

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial z^2} - [V(v) - V(v_f)]\frac{\partial \phi}{\partial z}. \tag{32}$$

So far, no approximation has been made so that Eq. (32) is exact and bears the same information as the initial Fokker-Planck equation (18). It is just written in a more convenient form which will allow us to examine the situation in the region of the front. Indeed, far from the front the profile is stationary (for sufficiently long time) and Eq. (32) is automatically satisfied as $\phi=1$ for $v \ll v_f(t)$ and $\phi=0$ for $v \gg v_f(t)$. If we consider values of the velocity that are close to $v_f(t)$, we can expand the term in brackets in Taylor series (the validity of this approximation will be studied in Sec. IV). Keeping only the first term in this expansion, we get

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial z^2} - g(t) z \frac{\partial \phi}{\partial z}, \tag{33}$$

where

$$g(t) = V'(v_f(t))\sqrt{D(v_f(t))}. \tag{34}$$

For future convenience, we set $\tau = 2t$ and define $h(\tau) = g(\tau/2)$. Therefore, the forgoing equation becomes

$$\frac{\partial \phi}{\partial \tau} = \frac{1}{2}\left(\frac{\partial^2 \phi}{\partial z^2} - h(\tau) z \frac{\partial \phi}{\partial z}\right). \tag{35}$$

The general solution of this equation, for an arbitrary initial condition, is given in Appendix A. Here, we look for particular solutions of the form $\phi(z,\tau) = \Phi(z/\chi(\tau))$. In Appendix A, we derive the condition under which such solutions describe the asymptotic long time behavior of the system (independently of the initial condition) and we check that this condition is fulfilled for all the explicit examples that we shall investigate in the following. To describe more general situations, one must use the results of Appendix A. Inserting the ansatz $\phi(z,\tau) = \Phi(z/\chi(\tau))$ in Eq. (35) leads to

$$\Phi'' + (2\chi\dot{\chi} - h\chi^2)x\Phi' = 0. \tag{36}$$

The variables of velocity and time separate provided that the term in parenthesis is a constant that we can arbitrarily set equal to

$$2\chi\dot{\chi} - h\chi^2 = 2. \tag{37}$$

Then, Φ is the function

$$\Phi(x) = \frac{1}{\sqrt{\pi}}\int_x^{+\infty} e^{-y^2} dy, \tag{38}$$

connected to the error function by $\Phi(x) = \frac{1}{2}(1 - \mathrm{erf}(x))$. It satisfies $\Phi(-\infty)=1$, $\Phi(0)=1/2$ and $\Phi(+\infty)=0$ so it reproduces the expected properties of the front (it can be seen as a smooth step function). On the other hand, solving for χ^2 in Eq. (37) and taking $\chi(1)=0$, we finally obtain

$$\chi^2(\tau) = 2\int_1^{\tau} e^{[H(\tau)-H(\tau')]} d\tau', \tag{39}$$

where H is a primitive of h with $H(1)=0$. The function $\phi(z,\tau) = \Phi(z/\chi(\tau))$ with (38) and (39) is the solution of Eq. (35) for all times, corresponding to the initial condition $\phi(z,1) = \eta(z)$ where η is a step function. It also governs the long time behavior of the system for a large class of initial conditions (Appendix A).

In particular, for $h(\tau) = \gamma/\tau$, we have

$$\frac{\partial \phi}{\partial \tau} = \frac{1}{2}\left(\frac{\partial^2 \phi}{\partial z^2} - \frac{\gamma}{\tau} z \frac{\partial \phi}{\partial z}\right), \tag{40}$$

and we recover the solution given in [15], namely

$$\phi(z,\tau) = \Phi\left[\frac{z}{\sqrt{2}}\left(\frac{1-\gamma}{\tau - \tau^{\gamma}}\right)^{1/2}\right]. \tag{41}$$

For $\gamma=-1$, a value that will frequently occur in the following examples, we have

$$\phi(z,\tau) = \Phi\left(\frac{z}{\tau^{1/2}}\right) \quad \text{for } \tau \gg 1. \tag{42}$$

In terms of the function g, these analytical solutions correspond to $g(t) = \lambda/t$. We come back to the original variables by setting $\tau = 2t$ and $\gamma = 2\lambda$.

IV. CONDITION OF VALIDITY

We shall now investigate in greater detail the ability of our approach to describe the front structure. First, we note that, within our approximations,

$$u(v_f(t),t) = \Phi(0) = \frac{1}{2}, \tag{43}$$

so that $v_f(t)$ gives the position of the half-height profile. On the other hand, noting that

$$V(v) - V(v_f) = V'(v_f)(v - v_f) + \frac{1}{2}V''(v_f)(v - v_f)^2 + \cdots \tag{44}$$

for $v \to v_f$, our approximation (33) will be valid in the range of velocities where we can neglect the second term in the Taylor expansion with respect of the first. This corresponds to

$$|v - v_f| \ll \left| \frac{2V'(v_f)}{V''(v_f)} \right|. \quad (45)$$

In this range of velocities, our approach is accurate. Now, it will provide a good description of the whole front if this range $|v-v_f|$ is larger than the typical front half-width $\Delta_f(t)/2$. This condition can be written

$$\Delta_f(t) \ll \left| \frac{4V'(v_f)}{V''(v_f)} \right|. \quad (46)$$

Now, the front width can be estimated by

$$\Delta_f(t) = \left| \frac{\partial u}{\partial v}(v_f(t),t) \right|^{-1}. \quad (47)$$

Within our approximation, this can be finally rewritten as

$$\Delta_f(t) = \sqrt{\pi}\chi(2t)\sqrt{D(v_f(t))}. \quad (48)$$

Therefore, our approach will provide a good description of the front structure if

$$\epsilon(t) \equiv \left| \frac{\sqrt{\pi}}{4} \frac{\chi(2t)\sqrt{D(v_f)}}{V'(v_f)/V''(v_f)} \right| \ll 1. \quad (49)$$

Since we shall be interested by the large time limit, it is particularly important to know the asymptotic behavior of the function $\epsilon(t)$ for $t \to +\infty$. This has to be considered case by case (see Sec. V).

The other assumption made in our study is that the long time behavior of the system is described by $\phi(z,\tau) = \Phi(z/\chi(\tau))$ with (38) and (39). In Appendix A, we show that this is the case if $\phi(z,1) \to 1$ for $z \to -\infty$, $\phi(z,1) \to 0$ for $z \to +\infty$ and if

$$H_2(\tau) = \int_1^\tau e^{-H(\tau')} d\tau' \to +\infty \quad (50)$$

for $\tau \to +\infty$. In particular, for $h(\tau) = \gamma/\tau$, we have

$$H(\tau) = \gamma \ln \tau, \quad H_1(\tau) = \frac{1}{\tau^{\gamma/2}}, \quad (51)$$

and

$$\chi^2(\tau) = \frac{2}{1-\gamma}(\tau - \tau^\gamma), \quad H_2(\tau) = \frac{\tau^{1-\gamma}-1}{1-\gamma}, \quad (52)$$

so that the criterion (50) is met if $\gamma \leqslant 1$ and not met if $\gamma > 1$.

V. PARTICULAR EXAMPLES

We shall now discuss explicitly some particular examples of physical interest and compare our theoretical predictions with direct numerical simulations of the Fokker-Planck equation.

A. Quadratic potential

We first consider the case of a quadratic potential $U(v) = v^2/2$ leading to the Kramers equation

$$\frac{\partial f}{\partial t} = \frac{1}{v^{d-1}} \frac{\partial}{\partial v}\left[v^{d-1} D(v) \left(\frac{\partial f}{\partial v} + fv \right) \right], \quad (53)$$

with a variable diffusion coefficient $D(v)$. If we assume a zero flux condition (19), then the stationary solutions are the Maxwellian distributions

$$f_e(v) = A e^{-v^2/2}. \quad (54)$$

We now consider various forms of diffusion coefficient and use the general theory developed in Sec. III to characterize the front profile and its evolution. We shall only give asymptotic expressions which are valid for large velocities and large time. This will be implicit in all the following calculations.

1. Power-law decay

For the diffusion coefficient

$$D(v) \sim v^{-\alpha}, \quad \alpha > 0, \quad (55)$$

we get

$$x \sim \frac{2}{\alpha+2} v^{(\alpha+2)/2}, \quad (56a)$$

$$v_f(t) \sim (\alpha t)^{1/\alpha}, \quad (56b)$$

and

$$g(t) \sim \frac{2-\alpha}{2\alpha}\frac{1}{t}, \quad \text{if } \alpha \neq 2,$$

$$g(t) \sim \frac{d-2}{2}\frac{1}{t^2}, \quad \text{if } \alpha = 2. \quad (57)$$

For $\alpha=2$, we have assumed that the subdominant corrections to the $D(v) \sim v^{-2}$ behavior are of order v^{-4} or smaller. Note that if $\alpha=2$ and $d=2$, then $g=0$ and the Fokker-Planck equation can be solved exactly (see Sec. VI).

For $\alpha \neq 2$, we find that $h = \gamma/\tau$ with $\gamma = 2/\alpha - 1$. The front profile is given by

$$u(v,t) \sim \Phi\left[\frac{v^{(\alpha+2)/2} - (\alpha t)^{(\alpha+2)/2\alpha}}{(\alpha+2)t^{1/2}} \left(\frac{1-\gamma}{1-(2t)^{\gamma-1}} \right)^{1/2} \right]. \quad (58)$$

The criterion (50) is fulfilled only for $\alpha \geqslant 1$, so that the above function provides the correct asymptotic behavior of the solution, for any initial condition, only in that case. For $\alpha < 1$, it can however provide the correct asymptotic behavior if the initial condition $\phi(z,1)$ is a step function (see Appendix A). Equation (58) returns the results obtained in [15]. Of course this formula is written for $\alpha \neq 1$, but it also provides the expression of the solution for $\alpha=1$ by passing to the limit $\alpha \to 1$ yielding

$$u(v,t) \sim \Phi\left[\frac{v^{3/2} - t^{3/2}}{3(t \ln(2t))^{1/2}} \right]. \quad (59)$$

On the other hand, for $\alpha=2$ we find that $h = 2(d-2)/\tau^2$ yielding $\chi^2(\tau) \sim 2\tau$ and $H_2(\tau) \sim \tau$ for $\tau \to +\infty$. We find that the

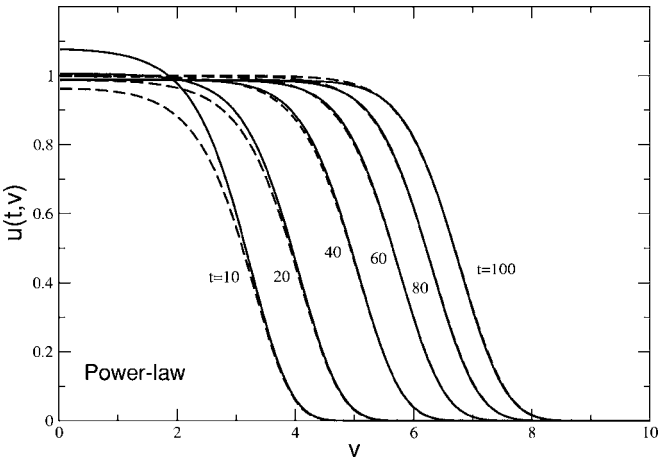

FIG. 1. Evolution of the front profile $u(v,t)$ for a power-law diffusion coefficient with $\alpha=3$. For sufficiently large times, we get a perfect agreement with the theoretical profile (58). Here and in the following, the straight lines correspond to the numerical simulation and the dashed lines to the theoretical prediction.

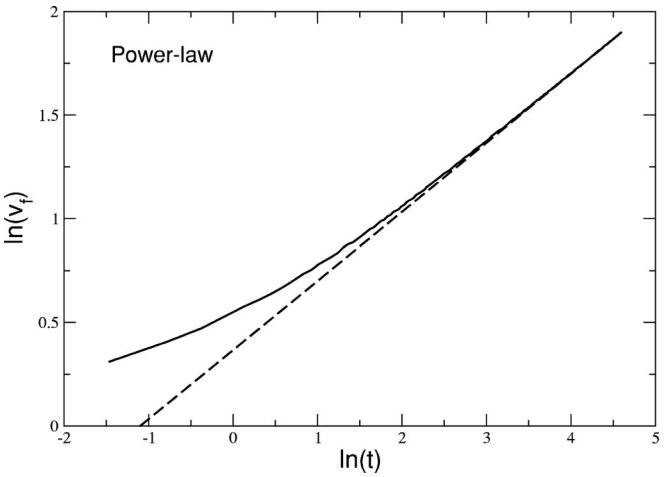

FIG. 2. Evolution of the front position $v_f(t)$ defined by Eq. (43) for a power-law diffusion coefficient with $\alpha=3$. For sufficiently large times, it coincides with the theoretical prediction (56b).

criterion (50) is satisfied and that the front profile is given by

$$u(v,t) \sim \Phi\left[\frac{v^2-2t}{4t^{1/2}}\right], \tag{60}$$

which turns out to be consistent with Eq. (58).

Concerning the validity of this approach in relation with the criterion (49), we have

$$\epsilon(t) \sim A_\alpha t^{-1/\alpha}, \quad \text{if } \alpha>1 \text{ and } \alpha\neq 2,$$

$$\epsilon(t) \sim \frac{B_\alpha}{t} \quad \text{if } \alpha<1,$$

$$\epsilon(t) \sim \frac{\sqrt{\pi}}{4}\frac{(\ln t)^{1/2}}{t} \quad \text{if } \alpha=1,$$

$$\epsilon(t) \sim \frac{3\sqrt{\pi}}{4}t^{-1/2} \quad \text{if } \alpha=2(d\neq 2), \tag{61}$$

where $A_\alpha=(\sqrt{2\pi}/8)\alpha^{1-1/\alpha}(\alpha-1)^{-1/2}$ and $B_\alpha=(\sqrt{\pi}/8)\alpha^{1-1/\alpha}(1-\alpha)^{-1/2}2^{1/\alpha-1/2}$. Therefore, the condition $\epsilon \ll 1$ is always fulfilled for sufficiently large times. For sake of illustration, we show the case $\alpha=3$ for $d=3$ (plasmas and stellar systems) in Figs. 1 and 2. These results are obtained by solving numerically the Fokker-Planck equation (53) starting from a step function: $f(v,t=0)=3/4\pi$ if $v<1$ and $f(v,t=0)=0$ if $v>1$ (water bag). In the simulation we have adopted the expression (7a) of the diffusion coefficient which reduces to Eq. (55) with $\alpha=3$ for $v\to+\infty$. Other examples are given in [15].

2. Gaussian decay

We now consider a diffusion coefficient of the form

$$D(v) \sim e^{-\gamma v^2}. \tag{62}$$

Considering the evolution of the high velocity tail, we have

$$x \sim \int_0^v e^{(\gamma/2)y^2}dy. \tag{63}$$

For large v, we obtain the relation (see Appendix B)

$$x \sim \frac{1}{\gamma v}e^{(\gamma/2)v^2}. \tag{64}$$

The position of the front $v_f(t)$ is determined by

$$\int^{\gamma v_f^2}\frac{e^y}{y}dy \sim 2(\gamma+1)t. \tag{65}$$

The integral can be expressed in terms of the exponential integral $E_i(x)$. For large times, we get

$$\frac{e^{\gamma v_f^2}}{\gamma v_f^2} \sim 2(\gamma+1)t. \tag{66}$$

To leading order, we have

$$v_f(t) \sim \left(\frac{\ln t}{\gamma}\right)^{1/2}. \tag{67}$$

We note that the evolution of the front is extremely slow (logarithmic). On the other hand, we find that

$$g(t) \sim -\frac{1}{2t}, \tag{68}$$

implying that the criterion (50) is fulfilled. In order to determine the front profile, we need first to evaluate $x_f(t)$. An equivalent for $t\to+\infty$ is obtained by combining Eqs. (64) and (66) leading to

$$x_f(t) \sim \sqrt{\frac{2(\gamma+1)t}{\gamma}}. \tag{69}$$

Therefore, the front profile is given by

$$u(v,t) \sim \Phi\left[\frac{\frac{1}{\gamma v}e^{(\gamma/2)v^2} - \sqrt{\frac{2(\gamma+1)t}{\gamma}}}{\sqrt{2t}}\right]. \qquad (70)$$

Concerning the validity of this approach in relation with the criterion (49), we find that $\epsilon(t) \to \frac{1}{4}[\pi\gamma/(\gamma+1)]^{1/2}$ for $t \to +\infty$ so that our approximations are marginally valid: $\epsilon(t)$ does not go to zero but it does not diverge with time neither.

We have performed numerical simulations of the Fokker-Planck equation (53) with the diffusion coefficient (62) with $\gamma=1/2$. As discussed in Sec. II, this equation describes the evolution of a "test particle" in a bath of "field particles" at statistical equilibrium (with Maxwellian distribution function $f_0 \sim e^{-v^2/2}$) for one-dimensional systems ($d=1$) with long-range interactions such as the HMF model. In Fig. 3, we show the "short" time evolution of the distribution function $f(v,t)$ starting from a step function: $f(v,0)=1/3$ for $v \leqslant 3$ whose width is larger than the thermal speed $v_{th}=1$ (dashed line). Since $D(v)$ rapidly decreases with the velocity for $v \gg v_{th}=1$, the high velocity component of the initial condition takes time to be depleted. Indeed, the core of the distribution rapidly reaches a Maxwellian distribution while the tail keeps the memory of the initial condition and remains flat for relatively long times. For the parameters of the simulation, this can even create a nonmonotonic distribution for short times as shown in Fig. 3 (see Appendix C).

In Fig. 4 we show the evolution of the normalized distribution function $u(v,t)$ for short and large times. In that case, we start from an initial condition: $f(v,0)=1$ if $v<1$ and $f(v,0)=0$ if $v>1$. In Fig. 5, we focus on the evolution of the front for large times and we compare the result of the numerical simulation with the theoretical prediction. In Fig. 6, we compare the evolution of the front displacement $v_f(t)$ with the theoretical prediction. The numerical results are in fair agreement with the theory although its domain of validity was expected to be marginal in that case according to our estimates. This fair agreement may be explained by the relatively small value of $\epsilon(+\infty)=\frac{1}{4}(\pi/3)^{1/2} \simeq 0.255\ldots$ for $\gamma=1/2$ even if this parameter does not strictly tends to zero for $t \to +\infty$. We note that for $t \to +\infty$ and $v \ll v_f(t)$, we have $u \sim \Phi(-\sqrt{(\gamma+1)/\gamma}) = \Phi(-\sqrt{\pi/4\epsilon})$ which has not converged to $u=1$ precisely because $\epsilon \neq 0$. However, for $\epsilon(+\infty) \simeq 0.255\ldots$, we find that $u \simeq 0.993$ which is very close to one.

In relation with the HMF model, we would like to recall that the present approach describes the relaxation of a given test particle immersed in a bath of field particles at statistical equilibrium. The relaxation is due to finite N effects (correlations). This does not describe the evolution of the distribution function of the system as a whole. The clearest reason is that the Fokker-Planck equation (1) resulting from a thermal bath approximation does not conserve energy contrary to the microcanonical evolution of a Hamiltonian system. In the

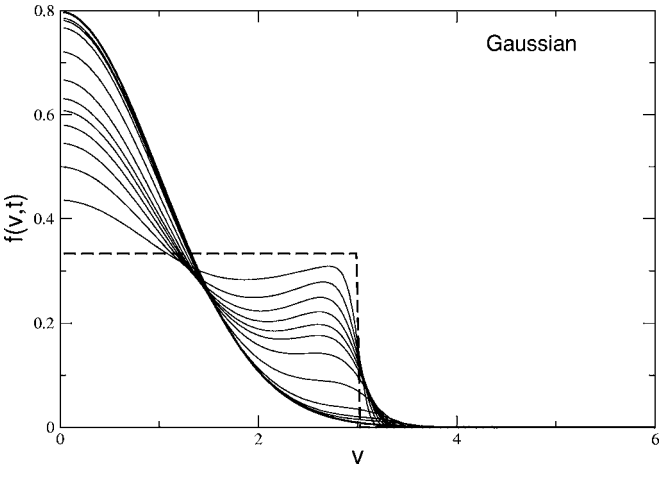

FIG. 3. Short time evolution of the distribution function $f(v,t)$ for a Gaussian diffusion coefficient. This figure shows the slow depletion of the high velocity tail due to the rapid decrease of the diffusion coefficient for $v \gg v_{th}=1$.

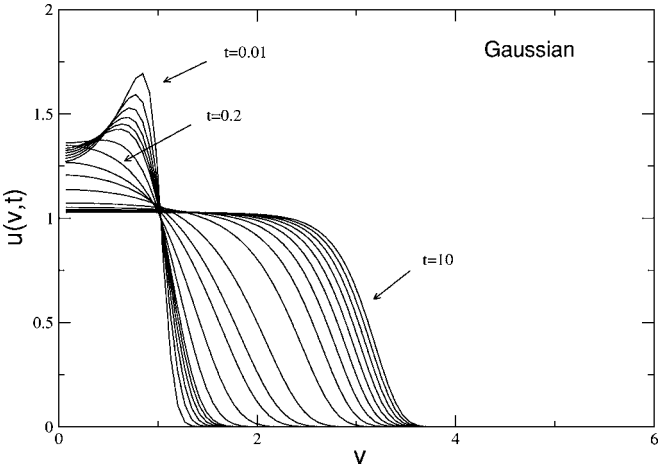

FIG. 4. Evolution of the normalized distribution function $u(v,t)$ for a Gaussian diffusion coefficient for short and large times. For short times the normalized distribution function forms a bump which slowly disappears as the core of the distribution becomes Maxwellian. For large times the function $u(v,t)$ has a front structure.

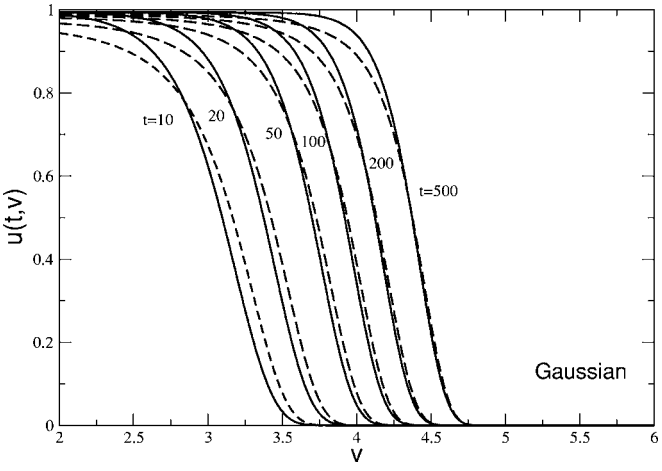

FIG. 5. Evolution of the front profile $u(v,t)$ for a Gaussian diffusion coefficient. For sufficiently large times, we get a fair agreement with the theoretical profile (70) although the validity of our approach was expected to be marginal in that case.

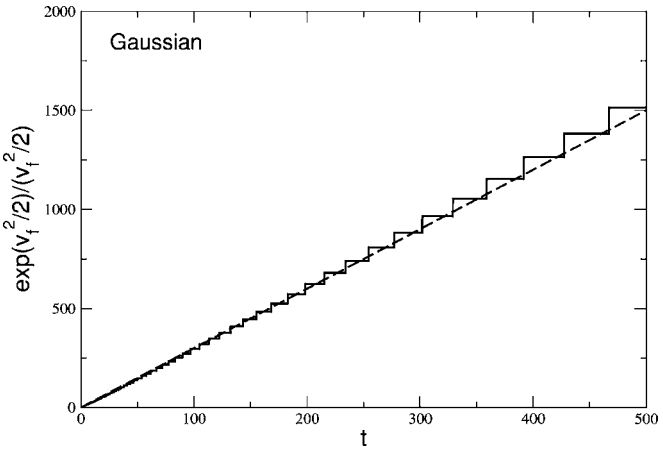

FIG. 6. Evolution of the front position $v_f(t)$ defined by Eq. (43) for a Gaussian diffusion coefficient. For sufficiently large times, we get a fair agreement with the theoretical prediction (66).

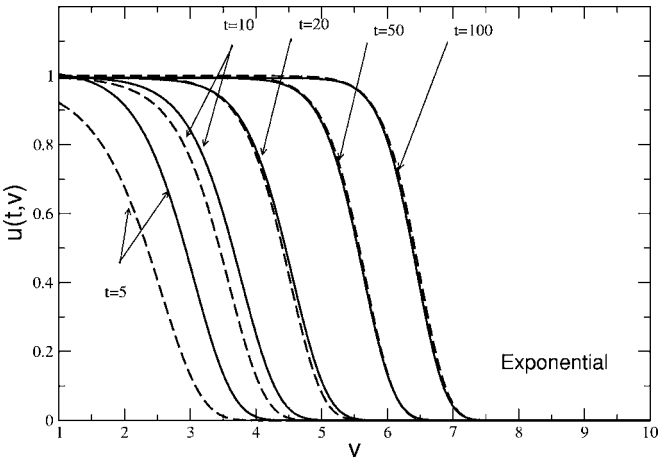

FIG. 7. Evolution of the front profile $u(v,t)$ for an exponential diffusion coefficient. For sufficiently large times, we get an excellent agreement with the theoretical profile (78).

collisionless regime, the distribution function $f(\theta,v,t)$ of the HMF model is governed by the Vlasov equation coupled to the mean-field potential $\Phi(\theta,t)=-(k/2\pi)\int\cos(\theta-\theta')f(\theta',v',t)d\theta'dv'$ produced by the particles. The Vlasov equation can experience a process of violent relaxation and converge toward a quasistationary state (QSS) on the coarse-grained scale [18]. Some dynamical theories of violent relaxation [16] based on a maximum entropy production principle (MEPP) propose to model the evolution of the coarse-grained distribution function by a generalized Fokker-Planck equation of the form (for a water-bag initial condition)

$$\frac{\partial\bar{f}}{\partial t}+v\frac{\partial\bar{f}}{\partial\theta}-\nabla\Phi\frac{\partial\bar{f}}{\partial v}=\frac{\partial}{\partial v}\left\{D(\theta,v,t)\left[\frac{\partial\bar{f}}{\partial v}+\beta(t)\bar{f}(\eta_0-\bar{f})v\right]\right\}, \tag{71}$$

where $\beta(t)$ evolves so as to conserve energy (see [16] for more details). An important point is that the diffusion coefficient is not constant but is related to the correlations of the fine-grained fluctuations. It can depend on position, velocity and time and can be very small in certain regions of phase space and for large times. The vanishing or smallness of the diffusion coefficient can slow down the dynamics and lead to a confinement of the distribution function in phase space which may be qualitatively similar to what is shown in Fig. 3. However, the dynamical equation (71) and its physical interpretation are different from Eq. (11), so that their similarity is, at most, an analogy.

3. Exponential decay

For the diffusion coefficient

$$D(v)\sim e^{-\gamma v}, \tag{72}$$

we get

$$x\sim\frac{2}{\gamma}e^{(\gamma/2)v}. \tag{73}$$

The evolution of the front is given by

$$\frac{e^{\gamma v_f}}{\gamma v_f}\sim t. \tag{74}$$

To leading order, we have

$$v_f(t)\sim\frac{1}{\gamma}\ln t, \tag{75}$$

which shows that the progression of the front is again very slow. We also have

$$g(t)\sim-\frac{1}{2t}, \tag{76}$$

implying that the criterion (50) is fulfilled. In order to determine the front profile, we need first to evaluate $x_f(t)$. An equivalent for $t\to+\infty$ is obtained by combining Eqs. (73) and (74) leading to

$$x_f(t)\sim\frac{2}{\gamma}(t\ln t)^{1/2}. \tag{77}$$

However, since the evolution with time is slow, we shall work with the more precise expression $x_f(t)\sim(2/\gamma)(t\ln(t\ln t))^{1/2}$ obtained by keeping the term of next order. With this expression, we find that the front profile is given by

$$u(v,t)\sim\Phi\left[\sqrt{2}\frac{e^{(\gamma/2)v}-(t\ln(t\ln t))^{1/2}}{\gamma t^{1/2}}\right]. \tag{78}$$

Concerning the validity of this approach in relation with the criterion (49), we find that $\epsilon(t)\sim(\gamma/8)\sqrt{2\pi}(\ln t)^{-1/2}$ for $t\to+\infty$ so that our approximations are valid for sufficiently large times. Note that the decay is slow with time (logarithmic) but the prefactor $(\gamma/8)\sqrt{2\pi}=0.313\ldots$ is relatively small (for $\gamma=1$) which can explain the very good agreement with the numerics even for moderate time scales. Figures 7 and 8 are obtained by solving the Fokker-Planck equation (53) with the diffusion coefficient (72) with $\gamma=1$, starting from an initial condition: $f(v,0)=1$ if $v<1$ and $f=0$ if $v>1$.

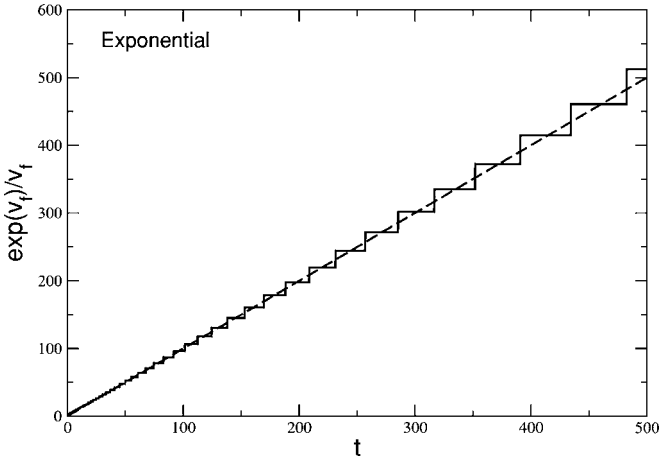

FIG. 8. Evolution of the front position $v_f(t)$ defined by Eq. (43) for an exponential diffusion coefficient. For sufficiently large times, we get an excellent agreement with the theoretical prediction (74).

4. Stretched exponential

For the diffusion coefficient

$$D(v) \sim e^{-\gamma v^{\delta}}, \quad \delta > 0, \tag{79}$$

we get

$$x \sim \int_0^v e^{(\gamma/2)y^{\delta}} dy. \tag{80}$$

We shall briefly discuss how the results depend on the value of δ. A more detailed analysis can be carried out as in the previous sections where the cases $\delta=1$ and $\delta=2$ are explicitly considered.

For $\delta<2$, the front evolution is given by

$$\frac{e^{\gamma v_f^{\delta}}}{\gamma v_f^{\delta}} \sim \delta t. \tag{81}$$

To leading order, we get

$$v_f(t) \sim \left(\frac{\ln t}{\gamma}\right)^{1/\delta}. \tag{82}$$

On the other hand,

$$g(t) \sim -\frac{1}{2t}, \tag{83}$$

implying that the criterion (50) is fulfilled. Concerning the validity of this approach in relation with the criterion (49), we find that

$$\epsilon(t) \sim \frac{\sqrt{2\pi\delta}}{8}\left(\frac{\ln t}{\gamma}\right)^{-(2-\delta)/2\delta}, \tag{84}$$

so that the criterion is satisfied for sufficiently large times (but slowly).

For $\delta>2$, the front evolution is given by

$$\frac{e^{\gamma v_f^{\delta}}}{(\gamma v_f^{\delta})^{2(\delta-1)/\delta}} \sim \frac{1}{2}\gamma^{2/\delta}\delta^2 t. \tag{85}$$

To leading order, we get

$$v_f(t) \sim \left(\frac{\ln t}{\gamma}\right)^{1/\delta}. \tag{86}$$

On the other hand,

$$g(t) \sim -\frac{1}{2t}, \tag{87}$$

implying that the criterion (50) is fulfilled. Concerning the validity of this approach in relation with the criterion (49), we find that $\epsilon \to \sqrt{\pi}/4$ so that this approach is marginally valid.

B. Linear potential

We now consider the case of a linear potential $U(v)=\gamma v$ leading to a Fokker-Planck equation of the form

$$\frac{\partial f}{\partial t} = \frac{1}{v^{d-1}}\frac{\partial}{\partial v}\left[v^{d-1}D(v)\left(\frac{\partial f}{\partial v} + \gamma f\right)\right]. \tag{88}$$

The stationary solution is

$$f_e(v) = Ae^{-\gamma v}. \tag{89}$$

For a diffusion coefficient decreasing algebraically with the velocity

$$D(v) \sim v^{-\alpha}, \quad \alpha > 0, \tag{90}$$

we get

$$x \sim \frac{2}{\alpha+2}v^{(\alpha+2)/2}, \quad v_f(t) \sim [(\alpha+1)\gamma t]^{1/(\alpha+1)}, \tag{91}$$

and

$$g(t) \sim -\frac{\alpha}{2(\alpha+1)}\frac{1}{t}, \tag{92}$$

implying that the criterion (50) is fulfilled. We also find that $\epsilon(t) \propto t^{-1/[2(\alpha+1)]}$ so that the validity criterion of our approach is always fulfilled for sufficiently large times.

C. Logarithmic potential

In this subsection, we consider the case of a constant diffusion coefficient $D(v)=1$ and a logarithmic potential $U(v)=(\alpha/2)\ln(1+v^2)$ in $d=1$ leading to a Fokker-Planck equation of the form

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial v} + \alpha f\frac{v}{1+v^2}\right). \tag{93}$$

This type of Fokker-Planck equations arises in the study of optical lattices [13,14]. The stationary solution of this equation is of the form

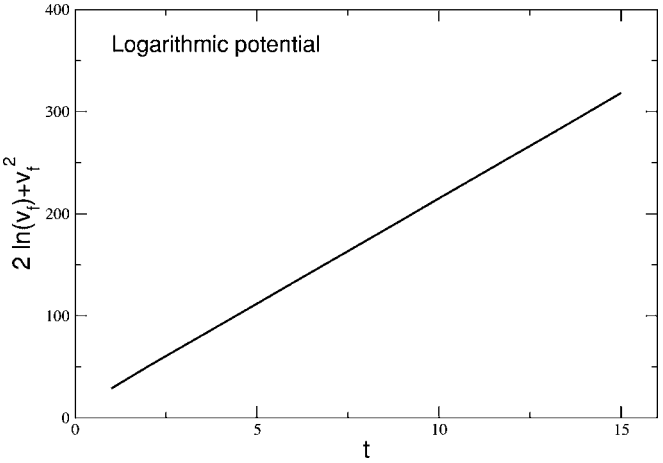

FIG. 9. Evolution of the front position $v_f(t)$ for a logarithmic potential with $\alpha=10$. We get a fair agreemment with the Lambert function.

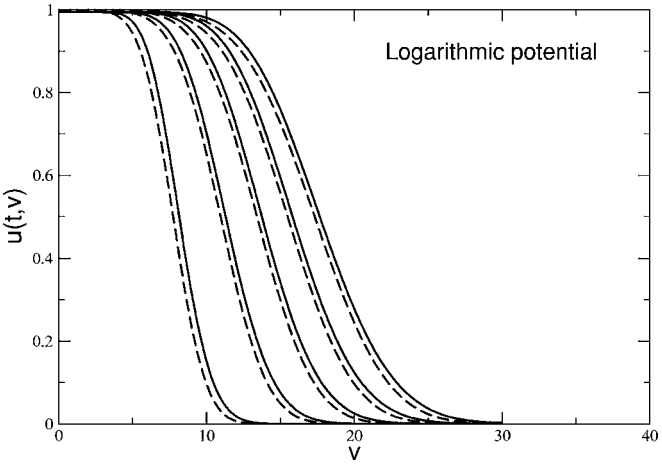

FIG. 10. Evolution of the front profile $u(v,t)$ for a logarithmic potential with $\alpha=10$.

$$f_e(v) = \frac{A}{(1+v^2)^{\alpha/2}}. \tag{94}$$

This is similar to a Tsallis distribution with $q=(\alpha-2)/\alpha$ [19]. However, it arises here from a linear Fokker-Planck equation (associated with the Boltzmann entropy) with a logarithmic potential, instead of a nonlinear Fokker-Planck equation (associated with the Tsallis entropy) with a quadratic potential [20–22]. The distribution (94) is normalizable provided that $\alpha>\alpha_{crit}=1$. The case $\alpha=2$ corresponds to the Lorentzian. The evolution of the front $v_f(t)$ satisfies

$$\frac{dv_f}{dt} = \frac{\alpha v_f}{1+v_f^2}. \tag{95}$$

The exact solution is given by $v_f^2 e^{v_f^2}=e^{2\alpha t+C}$ where C is a constant of integration. It can be written $v_f^2(t)=W(e^{2\alpha t+C})$ where $W(x)$ is the Lambert function which is solution of the transcendental equation $W(x)\exp[W(x)]=x$. For $t\to+\infty$, we get $v_f(t)\sim(2\alpha t)^{1/2}$ and $g(t)\sim -1/(2t)$. Therefore, the front profile is given by

$$u(v,t) = \Phi\left(\frac{v-\sqrt{2\alpha t}}{\sqrt{2t}}\right), \tag{96}$$

for large times. Concerning the validity of our approach with respect to the criterion (49), we find that $\epsilon\to(\pi/4\alpha)^{1/2}$ for $t\to+\infty$ so that our approach is marginally valid. The theoretical prediction is not very good for $\alpha=2$ but the agreement improves for large values of $\alpha\gg\alpha_{crit}=1$ for which ϵ is reduced. In particular, the results of numerical simulations performed with $\alpha=10$ are shown in Figs. 9 and 10. Concerning the evolution of the front, we find a good agreement with the Lambert function except that the measured exponent in Fig. 9 is ~ 20.6 instead of $2\alpha=20$. Therefore, the front increases a little bit faster than the theoretical prediction. This is confirmed in Fig. 10 which shows the evolution of the front profile. We note, however, the relatively good agreement between direct numerical simulation and theory. The slight discrepency is due to the finite value of $\epsilon\simeq 0.28$.

D. Fokker-Planck equations associated with one-dimensional systems with long-range interactions

We now consider the case of Fokker-Planck equations describing one dimensional systems with long-range interactions. In that case, the diffusion coefficient $D\sim f_0(v)$ for $v\to+\infty$ and the potential $U(v)=-\ln f_0(v)$ are expressed in terms of the distribution of the bath $f_0(v)$. The corresponding Fokker-Planck equation can be rewritten as (see Sec. II)

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v}\left(f_0\frac{\partial f}{\partial v} - f\frac{df_0}{dv}\right). \tag{97}$$

The general relations obtained in Sec. III take the simpler forms

$$\frac{dx}{dv}\sim\frac{1}{\sqrt{f_0(v)}},\quad V(v)\sim -\frac{3}{2}\frac{f_0'(v)}{\sqrt{f_0(v)}},\quad \frac{dv_f}{dt}\sim -\frac{3}{2}f_0'(v_f). \tag{98}$$

The case where $f_0(v)$ is a Gaussian distribution (thermal bath) has already been studied in Sec. V A 2. We shall consider other examples where $f_0(v)$ is not necessarily the statistical equilibrium state. This description still makes sense if f_0 is a stable stationary solution of the Vlasov equation because the relaxation of the system as a whole is slower than the relaxation of a test particle in a fixed distribution.

1. Exponential distribution

For the exponential distribution

$$f_0\sim e^{-\gamma v}, \tag{99}$$

we obtain

$$x\sim\frac{2}{\gamma}e^{(\gamma/2)v},\quad v_f\sim\frac{1}{\gamma}\ln\left(\frac{3}{2}\gamma^2 t\right), \tag{100}$$

and

$$g(t) \sim -\frac{1}{2t}, \tag{101}$$

implying that the criterion (50) is fulfilled. The front profile is given by

$$u(v,t) \sim \Phi\left[\sqrt{2}\frac{e^{(\gamma/2)v} - \left(\frac{3}{2}\gamma^2 t\right)^{1/2}}{\gamma t^{1/2}}\right]. \tag{102}$$

Concerning the validity of this approach in relation with the criterion (49), we find that $\epsilon \to \frac{1}{4}(\pi/3)^{1/2} \simeq 0.256\ldots$ for $t \to +\infty$ so that our approximations are marginally valid and are accurate since ϵ is relatively small.

2. Power-law distribution

For the power-law distribution

$$f_0 \sim v^{-\alpha}, \tag{103}$$

we obtain

$$x \sim \frac{2}{\alpha+2} v^{(\alpha+2)/2}, \quad v_f(t) \sim \left[\frac{3}{2}\alpha(\alpha+2)t\right]^{1/(\alpha+2)}, \tag{104}$$

and

$$g(t) \sim -\frac{1}{2t}, \tag{105}$$

implying that the criterion (50) is fulfilled. The front profile is given by

$$u(v,t) \sim \Phi\left[\sqrt{2}\frac{v^{(\alpha+2)/2} - \sqrt{\frac{3}{2}\alpha(\alpha+2)t}}{(\alpha+2)t^{1/2}}\right]. \tag{106}$$

Concerning the validity of this approach in relation with the criterion (49), we find that $\epsilon \to \frac{1}{8}(\alpha+4)\sqrt{4\pi/3\alpha(\alpha+2)}$ for $t \to +\infty$ so that our approximations are marginally valid.

VI. A CLASS OF EXACTLY SOLVABLE FOKKER-PLANCK EQUATIONS

In this section we give a class of Fokker-Planck equations (18) for which our approach turns out to be exact. More precisely, we derive a relationship between $D(v)$ and $U(v)$ under which the corresponding Fokker-Planck equation (18) can be exactly solved. First, we recall that Eqs. (18) and (32) are equivalent and no approximation has been made in the derivation of Eq. (32) from Eq. (18). However to pass from Eq. (32) to Eq. (33) for a general velocity field $V(v)$, we have made an approximation and have assumed that the concerned velocities are close enough to the position of the front $v_f(t)$. This is in fact the only approximation in the approach developed above. We now look for a situation where such an approximation is exact. This is the case if $V(v)=Ax+B$ where A and B are arbitrary constants. Using

$$\frac{d}{dx}[V(v)] = V'(v)\sqrt{D(v)} = A, \tag{107}$$

and replacing V by its expression (25), we equivalently get

$$\sqrt{D(v)}\frac{d}{dv}\left(\sqrt{D(v)}\left[U'(v) - \frac{d-1}{v} - \frac{1}{2}(\ln D)'(v)\right]\right) = A. \tag{108}$$

Defining $R(v)=\sqrt{D(v)}$, we obtain

$$R(v)\frac{d}{dv}\left[\left(U'(v) - \frac{d-1}{v}\right)R - R'(v)\right] = A. \tag{109}$$

If $U(v)$ and $R(v)=\sqrt{D(v)}$ satisfy Eq. (109), as discussed in Appendix D, we can use the results of Appendix A with $g(t)=A$ to obtain exact solutions of Eq. (18) for all times.

Let us make these solutions more explicit. First of all, for $h(\tau)=A$, we have

$$H(\tau) = A(\tau-1), \quad H_1(\tau) = e^{-(A/2)(\tau-1)}, \tag{110}$$

$$\chi^2(\tau) = \frac{2}{A}(e^{A(\tau-1)} - 1), \quad H_2(\tau) = \frac{1}{A}(1 - e^{-A(\tau-1)}). \tag{111}$$

We note that the criterion (50) is met only for $A \leqslant 0$. On the other hand, integrating

$$\frac{dx}{dt} = V(v) = Ax + B, \tag{112}$$

with $x_f(1/2)=0$, we get

$$x_f(t) = \frac{B}{A}(e^{(A/2)(2t-1)} - 1). \tag{113}$$

Then, substituting in Eq. (A27), we obtain the general solution

$$u(v,t) = \frac{1}{\sqrt{\frac{2\pi}{A}(1-e^{-2At})}} \times \int_{-\infty}^{+\infty} e^{-\{e^{-At}[x-(B/A)(e^{At}-1)]-y\}^2/(2/A)(1-e^{-2At})} u(y,0)dy, \tag{114}$$

where we have taken the origin of times at $t=0$. If we start from a step function $u(x,0)=\eta(x)$, the above expression reduces to

$$u(v,t) = \Phi\left[\frac{x - \frac{B}{A}(e^{At}-1)}{\sqrt{\frac{2}{A}}(e^{2At}-1)^{1/2}}\right]. \tag{115}$$

On the other hand, for $A=0$, we get

$$h(\tau) = 0, \quad H(\tau) = 0, \quad H_1(\tau) = 1, \quad H_2(\tau) = \tau - 1, \tag{116}$$

$$\chi^2(\tau) = 2(\tau - 1), \quad x_f(t) = \frac{B}{2}(2t - 1). \tag{117}$$

The general solution is

$$u(v,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-(x - Bt - y)^2/4t} u(y,0) dy, \tag{118}$$

and if we start from a step function $u(x,0) = \eta(x)$, we get

$$u(v,t) = \Phi\left(\frac{x - Bt}{2t^{1/2}}\right). \tag{119}$$

We recall that the above profiles are the exact solutions of the Fokker-Planck equation (18) when the functions $U(v)$ and $D(v)$ satisfy the differential equation (109). Let us examine some particular cases. If $A=0$, Eq. (109) can be integrated at once leading to the relation

$$D(v) = B^2 \left(\int_v^{+\infty} w^{d-1} e^{-U(w)} dw \right)^2 \frac{e^{2U(v)}}{v^{2(d-1)}}. \tag{120}$$

For the wide class of potentials $U(v)$ satisfying the following condition:

$$(d-1)\frac{U'(v)}{v} - U''(v) \ll U'(v)^2 \tag{121}$$

for large v, the behavior of the diffusion coefficient at infinity is given by

$$D(v) \sim \frac{C}{U'(v)^2}. \tag{122}$$

In particular, when $U(v) = v^2/2$ the diffusion coefficient $D(v)$ behaves like C/v^2 for $v \to +\infty$ and when $U(v) = \gamma v$ it tends to a constant. Note that condition (121) is not only satisfied by any nonconstant polynomial, but also by potentials $U(v)$ that dominate $\ln(v)$ for large velocities, as for instance $U(v) = [\ln(v)]^\alpha$, with $\alpha > 1$. Note finally that for $v \to 0$, the diffusion coefficient behaves like $1/v^{2(d-1)}$ which is divergent except for $d=1$.

In the case of a quadratic potential $U(v) = v^2/2$, the integral in Eq. (120) can be expressed in terms of the error function. There is a simplification in $d=2$ leading to $D(v) = 1/v^2$. This is precisely the case that we have found in Sec. V A. In that case, we have $x = v^2/2$ so that Eq. (118) can be explicitly written in terms of v. Inversely, for $D(v) = 1/v^2$ and $g=A$, we find from Eq. (109) that $U(v) = (A/8)v^4 + (B/2)v^2 + (d-2)\ln v$ so for this type of potential the solution of (114) applies with $x = v^2/2$. In the case of a linear potential $U(v) = \gamma v$, the integral in Eq. (120) can be expressed in terms of the incomplete Gamma function. There is a simplification in $d=1$ leading to $D(v) = 1$. In that case, we have $x = v$ so that Eq. (118) can be explicitly written in terms of v. Inversely, for $D(v) = 1$ and $g=A$, we find from Eq. (109) that $U(v) = (A/2)v^2 + Bv + (d-1)\ln v$ so for this type of potential the solution of (114) applies with $x = v$.

For $d=1$, $D(v) = 1$ and $U(v) = (A/2)v^2 + Bv$, we have $V(v) = Ax + B$ with $x = v$ and we are in the situation mentioned above. If we take $f(v,0) = \delta(v - v_0)$ and recall that $u(v,t) = f(v,t)/f_e(v)$, Eq. (114) gives after simplification

$$f(v,t) = \frac{1}{\sqrt{\frac{2\pi}{A}(1 - e^{-2At})}} \times e^{-[A/2(1-e^{-2At})][v - v_0 e^{-At} + (B/A)(1 - e^{-At})]^2}. \tag{123}$$

In particular, for $B=0$ and $A=1$, we recover the well-known solution [1]:

$$f(v,t) = \frac{1}{\sqrt{2\pi(1 - e^{-2t})}} e^{-(v - v_0 e^{-t})^2/2(1-e^{-2t})}. \tag{124}$$

In that case, $\langle v \rangle(t) = v_0 e^{-t}$. Alternatively, for $A=0$ and $B=1$, we get

$$f(v,t) = \frac{1}{\sqrt{4\pi t}} e^{-(v - v_0 + t)^2/4t}. \tag{125}$$

In that case, $\langle v \rangle(t) = v_0 - t$. Note that there is no normalizable stationary state for the Fokker-Planck equation (18) when $U(v) = Bv$ so that $f(v,t)$ tends to zero for large times and spreads so as to conserve the total mass. By contrast, when $A \neq 0$, the distribution function relaxes towards $f_e(v)$ for $t \to +\infty$. From Eq. (123), we can obtain the time evolution of the distribution function for any initial condition $f_0(v) = f(v,0)$ by multiplying Eq. (123) by $f_0(v_0)$ and integrating over v_0.

Note finally that for $d > 1$, the velocity $v = |\mathbf{v}|$ is restricted to positive values so that the preceding results must be slightly revised. The idea is to extend $f(v,t)$ by parity to negative values of v. We replace $D(v)$, $U(v)$, and v^{d-1} in Eq. (18) by $D(|v|)$, $U(|v|)$, and $|v|^{d-1}$. With this extension, Eq. (18) is invariant by the transformation $v \to -v$. Therefore, if $f(v,t_0)$ is even, $f(v,t)$ will remain even for all times and its values for $v \geqslant 0$ correspond to those of the distribution function that is a solution of the original Eq. (18). Thus we can apply the preceding results with almost no modification. We note however that since $V(v)$ and $x(v) = \int_0^v dw/\sqrt{D(w)}$ are odd, the case $V(v) = Ax + B$ is only possible if $B=0$. By contrast, in $d=1$, we can consider cases where $f(v,t)$, $D(v)$, and $U(v)$ have no special parity.

VII. CONCLUSION

In this paper, we have developed a general formalism to characterize the evolution of the distribution function tail for systems described by a Fokker-Planck equation with a diffusion coefficient and a friction force depending on the velocity. Our analytical results give good agreement with the numerics even in cases where the validity of our approach is marginal. When the diffusion coefficient decreases algebraically with the velocity, the progression of the front is also algebraic. When the diffusion coefficient decreases like an exponential or like a Gaussian, the progression of the front is logarithmic. The high velocity component of the distribution function keeps the memory of the initial condition for a long time and is slowly depleted. There are several applications of

our formalism to, e.g., stellar dynamics, plasma physics, vortex dynamics, the HMF model, optical lattices, etc. In future works, we shall study more specifically the dynamics of point vortices and investigate the evolution of the front profile and the time evolution of the correlation functions [8].

APPENDIX A: GENERAL SOLUTION OF EQ. (35)

In this appendix, we provide the general solution of the PDE,

$$2\frac{\partial \phi}{\partial \tau}=\frac{\partial^2 \phi}{\partial z^2}-h(\tau)z\frac{\partial \phi}{\partial z}, \tag{A1}$$

for an arbitrary initial condition $\phi_1(z)=\phi(z,1)$. Taking the Fourier transform of Eq. (A1) with the conventions

$$\phi(z)=\int \hat{\phi}(\xi)e^{i\xi z}d\xi, \quad \hat{\phi}(\xi)=\int \phi(z)e^{-i\xi z}\frac{dz}{2\pi}, \tag{A2}$$

and using the relation

$$\begin{aligned} z\frac{\partial \phi}{\partial z}&=\int zi\xi\hat{\phi}(\xi)e^{i\xi z}d\xi \\ &=\int \xi\hat{\phi}(\xi)\frac{\partial}{\partial \xi}(e^{i\xi z})d\xi \\ &=-\int \frac{\partial}{\partial \xi}[\xi\hat{\phi}(\xi)]e^{i\xi z}d\xi, \end{aligned} \tag{A3}$$

we get

$$2\frac{\partial \hat{\phi}}{\partial \tau}=[h(\tau)-\xi^2]\hat{\phi}+h(\tau)\xi\frac{\partial \hat{\phi}}{\partial \xi}. \tag{A4}$$

We introduce the change of variables

$$f(y,\tau)=\hat{\phi}(H_1(\tau)y,\tau), \quad \xi=H_1(\tau)y \tag{A5}$$

and choose the function $H_1(\tau)$ such that

$$\frac{H_1'(\tau)}{H_1(\tau)}=-\frac{h(\tau)}{2}. \tag{A6}$$

Substituting Eq. (A5) in Eq. (A4), we find that $f(y,\tau)$ satisfies

$$\frac{\partial f}{\partial \tau}+\frac{1}{2}[H_1^2(\tau)y^2-h(\tau)]f=0. \tag{A7}$$

Let $H(\tau)$ be the primitive of $h(\tau)$ such that

$$H(\tau)=\int_1^\tau h(\tau')d\tau'. \tag{A8}$$

Then, we choose H_1, solution of Eq. (A6), such that

$$H_1(\tau)=e^{-H(\tau)/2}. \tag{A9}$$

By convention, $H(1)=0$ and $H_1(1)=1$. Equation (A7) can be integrated leading to

$$f(y,\tau)=f(y,1)e^{H(\tau)/2}e^{-(1/2)H_2(\tau)y^2}, \tag{A10}$$

where we have defined

$$H_2(\tau)=\int_1^\tau H_1(\tau')^2d\tau'=\int_1^\tau e^{-H(\tau')}d\tau'. \tag{A11}$$

Returning to original variables, we obtain

$$\hat{\phi}(\xi,\tau)=\hat{\phi}_1\left(\frac{\xi}{H_1(\tau)}\right)e^{H(\tau)/2}e^{-(1/2)[H_2(\tau)/H_1^2(\tau)]\xi^2}. \tag{A12}$$

We now observe that

$$\frac{H_2(\tau)}{H_1^2(\tau)}=\int_1^\tau e^{H(\tau)-H(\tau')}d\tau'\equiv\frac{1}{2}\chi^2(\tau), \tag{A13}$$

where $\chi(\tau)$ is the function introduced in Eq. (39). Therefore the general solution of Eq. (A1) in Fourier space can be written

$$\hat{\phi}(\xi,\tau)=\frac{1}{H_1(\tau)}\hat{\phi}_1\left(\frac{\xi}{H_1(\tau)}\right)e^{-(1/4)\chi^2(\tau)\xi^2}. \tag{A14}$$

Defining

$$q(z)=\phi_1(H_1(\tau)z)\leftrightarrow\hat{q}(\xi)=\frac{1}{H_1(\tau)}\hat{\phi}_1\left(\frac{\xi}{H_1(\tau)}\right), \tag{A15}$$

$$g(z)=G(z/\chi(\tau))\leftrightarrow\hat{g}(\xi)=\chi(\tau)\hat{G}(\chi(\tau)\xi), \tag{A16}$$

where

$$G(z)=e^{-z^2}\leftrightarrow\hat{G}(\xi)=\frac{1}{2\sqrt{\pi}}e^{-\xi^2/4}, \tag{A17}$$

we can rewrite Eq. (A14) in the form

$$\hat{\phi}(\xi,\tau)=\frac{2\sqrt{\pi}}{\chi(\tau)}\hat{q}(\xi)\hat{g}(\xi). \tag{A18}$$

Taking the inverse Fourier transform, we can express the solution of Eq. (A1) as a convolution

$$\phi(z,\tau)=\frac{2\sqrt{\pi}}{\chi(\tau)}\int q(z-z')g(z')\frac{dz'}{2\pi}, \tag{A19}$$

or, equivalently

$$\phi(z,\tau)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-x^2}\phi_1(H_1(\tau)[z-\chi(\tau)x])dx. \tag{A20}$$

By direct substitution, we can check that Eq. (A20) is indeed solution of Eq. (A1).

If $\phi_1(z)=\eta(z)$ is a step function with $\eta(z)=1$ for $z<0$ and $\eta(z)=0$ for $z>0$, we immediately find that

$$\phi(z,\tau)=\frac{1}{\sqrt{\pi}}\int_{z/\chi(\tau)}^{+\infty}e^{-x^2}dx=\Phi\left(\frac{z}{\chi(\tau)}\right), \tag{A21}$$

and we recover the result of Sec. III. Then, the general solution of Eq. (A1) can be put in the form

$$\phi(z,\tau)=\Phi\left(\frac{z}{\chi(\tau)}\right)+\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-x^2}(\phi_1-\eta)\times(H_1(\tau)[z-\chi(\tau)x])dx. \quad \text{(A22)}$$

We introduce the new variable $z'=z/\chi(\tau)$ and consider the limit $t\to+\infty$. The integral depending on the initial condition is given by

$$I=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-x^2}(\phi_1-\eta)[\sqrt{2H_2(\tau)}(z'-x)]dx. \quad \text{(A23)}$$

We shall assume that $\phi_1(z)\to 1$ for $z\to-\infty$ and $\phi_1(z)\to 0$ for $z\to+\infty$. Then, if $H_2(\tau)\to+\infty$ for $\tau\to+\infty$, the function $(\phi_1-\eta)[\sqrt{2H_2(\tau)}(z'-x)]$ will be very peaked around $x=z'$ and we can approximate the integral by

$$I\sim\frac{e^{-z'^2}}{\sqrt{2\pi H_2(\tau)}}\int_{-\infty}^{+\infty}(\phi_1-\eta)(x)dx\to 0. \quad \text{(A24)}$$

Therefore, the condition that the solution of Eq. (A1) tends asymptotically to the function (A21) for $\tau\to+\infty$ is that $(\phi_1-\eta)(\pm\infty)=0$ and

$$H_2(\tau)=\int_1^\tau e^{-H(\tau')}d\tau'\to+\infty \quad \text{(A25)}$$

for $\tau\to+\infty$. We can also write the general solution (A20) in the form

$$\phi(z,\tau)=\frac{1}{\sqrt{2\pi H_2(\tau)}}\int_{-\infty}^{+\infty}e^{-[H_1(\tau)z-x]^2/2H_2(\tau)}\phi_1(x)dx. \quad \text{(A26)}$$

Returning to original variables, we get

$$u(v,t)=\frac{1}{\sqrt{2\pi H_2(2t)}}\int_{-\infty}^{+\infty}e^{-(H_1(2t)[x-x_f(t)]-y)^2/2H_2(2t)}u(y,1/2)dy, \quad \text{(A27)}$$

where $x=x(v)$ and we have taken by convention $x_f(t=1/2)=0$.

APPENDIX B: ASYMPTOTIC BEHAVIOR OF EQ. (63)

Setting $z=(\gamma/2)y^2$, we can rewrite Eq. (63) in the equivalent form

$$x\sim\frac{1}{\sqrt{2\gamma}}e^{(\gamma/2)v^2}\int_0^{(\gamma/2)v^2}\frac{e^{z-(\gamma/2)v^2}}{\sqrt{z}}dz. \quad \text{(B1)}$$

With the change of variables $y=-z+(\gamma/2)v^2$, we get

$$x\sim\frac{1}{\gamma v}e^{(\gamma/2)v^2}\int_0^{(\gamma/2)v^2}\frac{e^{-y}}{\sqrt{1-\frac{2y}{\gamma v^2}}}dy. \quad \text{(B2)}$$

Then taking the limit $v\to+\infty$, we find that

$$x\sim\frac{1}{\gamma v}e^{(\gamma/2)v^2}\int_0^{+\infty}e^{-y}dy\sim\frac{1}{\gamma v}e^{(\gamma/2)v^2}. \quad \text{(B3)}$$

APPENDIX C: CRITERION FOR THE MONOTONICITY OF $f(v,t)$

In this appendix, we establish a criterion which guarantees the monotonicity of $f(v,t)$ for all times, if the distribution function is initially monotonic (decreasing). Let us first rewrite the Fokker-Planck equation (18) in the form

$$\frac{\partial f}{\partial t}=D\frac{\partial^2 f}{\partial v^2}+\left(D'+DU'+\frac{d-1}{v}D\right)\frac{\partial f}{\partial v}+\left[(DU')'+\frac{d-1}{v}DU'\right]f. \quad \text{(C1)}$$

Because of the positivity of the diffusion coefficient, a classical comparison principle states that if $f(v,t)$ is initially positive, it will remain positive for all times [23]. The idea is now to apply the same argument to $g=\partial f/\partial v$. Taking the derivative of Eq. (C1) with respect to v, we get

$$\frac{\partial g}{\partial t}=D\frac{\partial^2 g}{\partial v^2}+\left(2D'+DU'+\frac{d-1}{v}D\right)\frac{\partial g}{\partial v}+\left[D''+2(DU')'+\frac{d-1}{v}DU'+(d-1)\left(\frac{D}{v}\right)'\right]g+\left[(DU')''+(d-1)\left(\frac{DU'}{v}\right)'\right]f. \quad \text{(C2)}$$

If

$$(DU')''+(d-1)\left(\frac{DU'}{v}\right)'\leqslant 0, \quad \text{(C3)}$$

then

$$\frac{\partial g}{\partial t}\leqslant D\frac{\partial^2 g}{\partial v^2}+\left(2D'+DU'+\frac{d-1}{v}D\right)\frac{\partial g}{\partial v}+\left[D''+2(DU')'+\frac{d-1}{v}DU'+(d-1)\left(\frac{D}{v}\right)'\right]g. \quad \text{(C4)}$$

Again we use a comparison principle and deduce from this inequality that if $g(v,t)$ is initially negative, it will remain negative for all times. Therefore, if $f(v,t)$ is initially decreasing, it will remain decreasing for all times if the criterion (C3) is satisfied. This criterion is just a *sufficient* condition of monotonicity.

Let us give some particular examples of application. In the case of a quadratic potential $U(v)=v^2/2$, the criterion (C3) becomes

$$(Dv)''+(d-1)D'\leqslant 0. \quad \text{(C5)}$$

For a diffusion coefficient of the form $D=v^{-\alpha}$, we find the condition $\alpha\leqslant d$. In particular, for the case $\alpha=d=3$, the criterion is satisfied. We can check that the criterion (C5) is also

satisfied for all v for the diffusion coefficient (7a) corresponding to the Coulombian (or Newtonian) potential of interaction. Alternatively, for a diffusion coefficient of the form $D=e^{-\gamma v^2}$, we find that the criterion (C5) is not satisfied for large values of v [more precisely $v>(d+\sqrt{d^2+8\gamma})/4\gamma$]. Therefore, the profile $f(v,t)$ can be nonmonotonic even if the initial condition is monotonic, as in the case of Fig. 3.

APPENDIX D: GENERAL SOLUTION OF EQ. (109)

In this appendix, we show that Eq. (109) can be solved explicitly. First of all, we note that it can be written

$$\left(U'(v)-\frac{d-1}{v}\right)R-R'(v)=Ax+B. \tag{D1}$$

Since $R=dv/dx$, we obtain

$$\left(U'(v)-\frac{d-1}{v}-\frac{R'(v)}{R(v)}\right)dv=(Ax+B)dx, \tag{D2}$$

which leads to

$$e^{-U(v)}v^{d-1}R(v)=Ke^{-[(A/2)x^2+Bx]}, \tag{D3}$$

where K is a constant. Equation (D3) can again be integrated into

$$\int_v^{+\infty}e^{-U(w)}w^{d-1}dw=K\int_{x(v)}^{+\infty}e^{-[(A/2)y^2+By]}dy. \tag{D4}$$

For $A=0$, we obtain

$$\int_v^{+\infty}e^{-U(w)}w^{d-1}dw=\frac{K}{B}e^{-Bx}. \tag{D5}$$

Substituting this relation in Eq. (D3), we recover the result of Eq. (120). On the other hand, for $A\neq 0$, we get

$$\int_v^{+\infty}e^{-U(w)}w^{d-1}dw=K\sqrt{\frac{2\pi}{A}}e^{B^2/2A}\Phi\left[\sqrt{\frac{A}{2}}\left(\frac{B}{A}+x\right)\right], \tag{D6}$$

where $\Phi(x)$ is defined in Eq. (38). Substituting the foregoing relation in Eq. (D3), we find that

$$D(v)=\frac{e^{2U(v)}}{v^{2(d-1)}}F\left[\int_v^{+\infty}Ce^{-U(w)}w^{d-1}dw\right], \tag{D7}$$

where

$$F(x)=\exp\{-2(\Phi^{-1}(x))^2\}, \tag{D8}$$

and C is a constant.

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